

Announcements

1) Turfe Lecture - general audience math talk, Wednesday next week, 3-4, CB 1030

Stephen DeBacker:

Connections between geometry and number theory

2) New Matlab resource added - beginner's guide to programming with Matlab

Algorithm for Back Substitution

$$\text{Solving } Rx = y = Q^T b$$

for x where

$$A = QR \in \mathbb{C}^{m \times m} \text{ and}$$

A is invertible.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$R =$ upper triangular, $m \times m$

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,m} \\ 0 & r_{2,2} & r_{2,3} & \dots & r_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & r_{n-1,m-1} & r_{n-1,m} \\ 0 & \dots & \dots & 0 & r_{m,m} \end{bmatrix}$$

Apply R to x .

(Note A invertible $\Rightarrow r_{j,j} \neq 0$
 $\forall (1 \leq j \leq m)$)

$$R_x =$$

$$\begin{bmatrix} r_{1,1} & r_{1,2} & \dots & & r_{1,m} \\ 0 & r_{2,2} & r_{2,3} & \dots & r_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & & & r_{m-1,m-1} & r_{m-1,m} \\ 0 & & & & r_{m,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^m r_{1,j} x_j \\ \sum_{j=2}^m r_{2,j} x_j \\ \vdots \\ r_{m-1,m-1} x_{m-1} + r_{m-1,m} x_m \\ r_{m,m} x_m \end{bmatrix}$$

We solve

$Rx = y$, which is

$$\begin{bmatrix} \sum_{j=1}^m r_{1,j} x_j \\ \sum_{j=2}^m r_{2,j} x_j \\ \vdots \\ r_{m-1,m-1} x_{m-1} + r_{m-1,m} x_m \\ r_{m,m} x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix}.$$

Solve from bottom-up.

We have:

$$1) \quad \Gamma_{m,m} X_m = y_m, \text{ so}$$

$$X_m = \frac{y_m}{\Gamma_{m,m}}$$

$$2) \quad \Gamma_{m-1,m-1} X_{m-1} + \Gamma_{m-1,m} X_m = y_{m-1}$$

$$X_{m-1} = \frac{y_{m-1} - \Gamma_{m-1,m} X_m}{\Gamma_{m-1,m-1}}$$

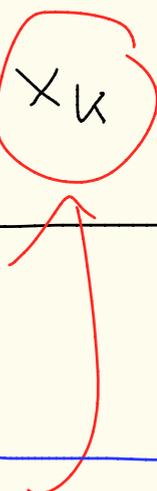
$$\begin{aligned}
 3) \quad & \Gamma_{m-2, m-2} X_{m-2} \\
 & + \Gamma_{m-2, m-1} X_{m-1} \\
 & + \Gamma_{m-2, m} X_m \\
 & = y_{m-2}, \quad \text{so}
 \end{aligned}$$

$$X_{m-2} = y_{m-2} \Gamma_{m-2, m-1} X_{m-1} \Gamma_{m-2, m} X_m$$

Substitute solution
from 2)

Substitute
solution
from 1)

$m - j + 1$)

$$x_j = \frac{y_j - \sum_{k=j+1}^m r_{jk} x_k}{r_{jj}}$$


Substitute
in from previous
steps

$(1 \leq j \leq m)$

Book painfully checks this
is backwards stable - that
is the only thing that
is (almost) thoroughly
checked in this text!

We won't do it.

Programming with Matlab

"For" loops :

Calculate the trace
of an arbitrary square
matrix using a "for"
loop .

Back to Least Squares

Recall: $A \in \mathbb{C}^{m \times n}$, rank n

"Solve" $Ax = b$.

No problem when $b \in \text{ran}(A)$.

If not, we use least squares as follows:

Look at minimizing

$$\|Ax - b\|_2.$$

If $b \in \text{ran}(A)$, the minimum is zero.

If not, the minimum occurs when Ax is the orthogonal projection of b onto $\text{ran}(A)$.

Recall: normal
equations

$$A^* A x = A^* b$$

Since A is rank n ,

$A^* A$ is invertible,

and we get

$$x = (A^* A)^{-1} A^* b$$

$= A^+$, the pseudoinverse

Then

$$y = Ax = AA^+ b$$

You can check that

$AA^+ = P$, the orthogonal projection onto $\text{ran}(A)$.