Announcements

1) Turfe Lecture – general audience math talk, Wednesday next week, 3-4, CB 1030
Stephen DeBacker:
connections between geometry and number theory

2) New Matlab resource added – beginner's guide to programming with Matlab

Algorithm for Back Substitution

Solving $Rx = y = Q^* b$

for x where

$A = QR \in \mathbb{C}^{m \times m}$ and

A is invertible.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$R =$ upper triangular, $m \times m$

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & \cdots & r_{1,m} \\ 0 & r_{2,2} & r_{2,3} & \cdots & r_{2,m} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & r_{m-1,m-1} & r_{m-1,m} \\ 0 & \cdots & \cdots & 0 & r_{m,m} \end{bmatrix}$$

Apply R to x.

(Note A invertible $\Rightarrow r_{j,j} \neq 0$ $\forall \ 1 \leq j \leq m$)

$$Rx =$$

$$\begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & \cdots & r_{1,m} \\ 0 & r_{2,2} & r_{2,3} & \cdots & r_{2,m} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & r_{m-1,m-1} & r_{m-1,m} \\ 0 & \cdots & \cdots & 0 & r_{m,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{m} r_{1,j} x_j \\ \sum_{j=2}^{m} r_{2,j} x_j \\ \vdots \\ r_{m-1,m-1} x_{m-1} + r_{m-1,m} x_m \\ r_{m,m} x_m \end{bmatrix}$$

We solve

$Rx = y$, which is

$$\begin{bmatrix} \sum_{j=1}^{m} r_{1,j} x_j \\ \sum_{j=2}^{m} r_{2,j} x_j \\ \vdots \\ r_{m-1,m-1} x_{m-1} + r_{m-1,m} x_m \\ r_{m,m} x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{bmatrix}.$$

Solve from bottom-up.

We have:

1) $r_{m,m} x_m = y_m$, so

$$x_m = \frac{y_m}{r_{m,m}}$$

2) $r_{m-1,m-1} x_{m-1} + r_{m-1,m} x_m$

$= y_{m-1}$

$$x_{m-1} = \frac{y_{m-1} - r_{m-1,m} x_m}{r_{m-1,m-1}}$$

3) $r_{m-2,m-2} x_{m-2}$

$+ r_{m-2,m-1} x_{m-1}$

$+ r_{m-2,m} x_m$

$= y_{m-2}$, so

$$x_{m-2} = \frac{y_{m-2} - r_{m-2,m-1} x_{m-1} - r_{m-2,m} x_m}{r_{m-2,m-2}}$$

Substitute solution from 2)

Substitute solution from 1)

$m-j+1)$

$$\boxed{x_j = \frac{y_j - \sum_{k=j+1}^{m} r_{j,k} x_k}{r_{j,j}}}$$

Substitute in from previous steps

$(1 \le j \le m)$

Book painfully checks this is backwards stable – that is the only thing that is (almost) thoroughly checked in this text!

We won't do it.

Programming with Matlab

"For" loops:

Calculate the trace of an arbitrary square matrix using a "for" loop.

Back to Least Squares

Recall: $A \in \mathbb{C}^{m \times n}$, rank n

"Solve" $Ax = b$.

No problem when $b \in \operatorname{ran}(A)$.

If not, we use least squares as follows:

Look at minimizing

$\|Ax - b\|_2$.

If $b \in \text{ran}(A)$, the minimum is zero.

If not, the minimum occurs when Ax is the orthogonal projection of b onto $\text{ran}(A)$.

Recall: normal equations

$$A^*Ax = A^*b$$

Since A is rank n, A^*A is invertible, and we get

$$x = \underbrace{(A^*A)^{-1}A^*}_{=A^+\text{, the pseudoinverse}} b$$

Then

$$y = Ax = AA^{+}b$$

You can check that

$AA^{+} = P$, the orthogonal projection onto $\operatorname{ran}(A)$.